6. Mark the coordinates of all points where the circle crosses the axes on your diagram. Label these points as P(θ) = (*x*, *y*), where P(θ) represents a point on the circle that has a central angle θ in standard position. For example, label the point where the circle crosses the positive *y*-axis as $P\left(\frac{\pi}{2}\right) = (0, 1)$.

7. Now, create a second number line. Label the ends as 0 and 2π. Divide this number line into 12 equal segments. Label divisions in terms of π. Express fractional values in lowest terms.

Reflect and Respond

8. Since each number line shows the circumference of the can and the circle to be 2π units, what assumption is being made about the length of the radius?

9. **a)** Two students indicate that the points in step 6 are simply multiples of $\frac{\pi}{2}$. Do you agree? Explain.

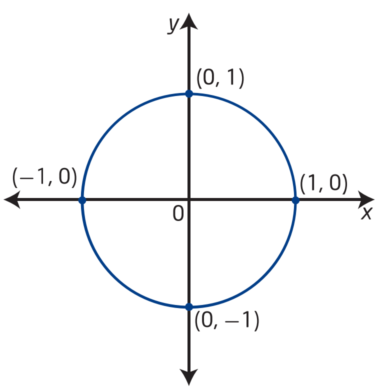
   b) In fact, they argue that the values on the original number line are all multiples of $\frac{\pi}{4}$. Is this true? Explain.

10. Show how to determine the coordinates for $P\left(\frac{\pi}{4}\right)$. Hint: Use your knowledge of the ratios of the side lengths of a 45°-45°-90° triangle. Mark the coordinates for all the points on the circle that are midway between the axes. What is the only difference in the coordinates for these four points? What negative values for θ would generate the same points on the circle midway between the axes?

Link the Ideas

Unit Circle

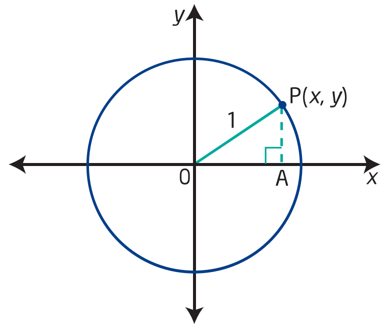
The circle you drew in the investigation is a **unit circle**.



unit circle
- a circle with radius 1 unit
- a circle of radius 1 unit with centre at the origin on the Cartesian plane is known as *the* unit circle

You can find the equation of the unit circle using the Pythagorean theorem.

Consider a point P on the unit circle. Let P have coordinates (x, y). Draw right triangle OPA as shown.



$OP = 1$ — The radius of the unit circle is 1.

$PA = |y|$ — The absolute value of the y-coordinate represents the distance from a point to the x-axis. Why is this true?

$OA = |x|$

$(OP)^2 = (OA)^2 + (PA)^2$ — Pythagorean theorem

$1^2 = |x|^2 + |y|^2$

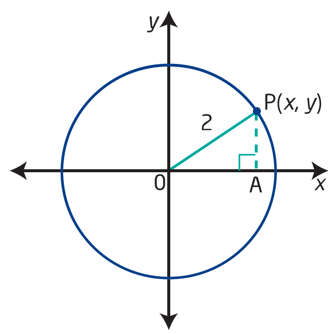
$1 = x^2 + y^2$ — How would the equation for a circle with centre O(0, 0) differ if the radius were r rather than 1?

The equation of the unit circle is $x^2 + y^2 = 1$.

Example 1

Equation of a Circle Centred at the Origin

Determine the equation of the circle with centre at the origin and radius 2.



Solution

Choose a point, P, on the circle with coordinates (x, y).

The radius of the circle is 2, and a vertical line from the y-coordinate to the x-axis forms a right angle with the axis. This means you can use the Pythagorean theorem.

$|x|^2 + |y|^2 = 2^2$
$x^2 + y^2 = 4$

Since this is true for every point P on the circle, the equation of the circle is $x^2 + y^2 = 4$.

Your Turn

Determine the equation of a circle with centre at the origin and radius 6.

Example 2

Determine Coordinates for Points of the Unit Circle

Determine the coordinates for all points on the unit circle that satisfy the conditions given. Draw a diagram in each case.

a) the x-coordinate is $\frac{2}{3}$

b) the y-coordinate is $-\frac{1}{\sqrt{2}}$ and the point is in quadrant III

Solution

a) Coordinates on the unit circle satisfy the equation $x^2 + y^2 = 1$.

$$\left(\frac{2}{3}\right)^2 + y^2 = 1$$

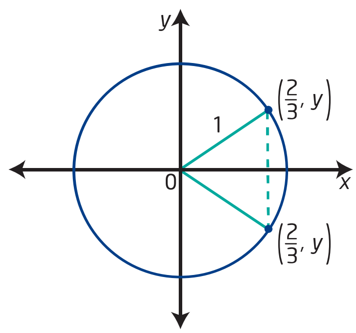
Since x is positive, which quadrants could the points be in?

$$\frac{4}{9} + y^2 = 1$$

$$y^2 = \frac{5}{9}$$

$$y = \pm\frac{\sqrt{5}}{3}$$

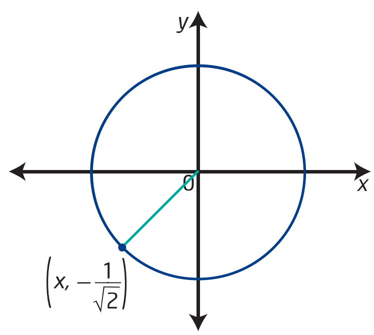
Why are there two answers?



Two points satisfy the given conditions: $\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$ in quadrant I and $\left(\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$ in quadrant IV.

b) $y = -\frac{1}{\sqrt{2}}$

y is negative in quadrants III and IV. But the point is in quadrant III, so x is also negative.



$$x^2 + y^2 = 1$$

$$x^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$$

$$x^2 + \frac{1}{2} = 1$$

$$x^2 = \frac{1}{2}$$

$$x = -\frac{1}{\sqrt{2}}$$

Why is there only one answer?

The point is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, or $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Your Turn

Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram and tell which quadrant(s) the points lie in.

a) $\left(-\frac{5}{8}, y\right)$

b) $\left(x, \frac{5}{13}\right)$, where the point is in quadrant II

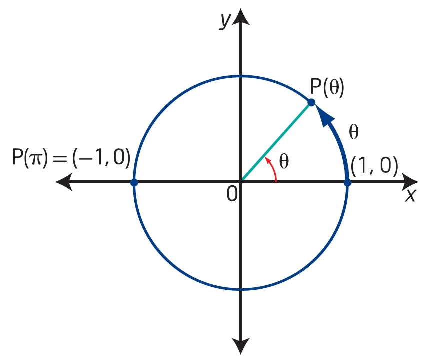
Relating Arc Length and Angle Measure in Radians

The formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the radius, applies to any circle, as long as a and r are measured in the same units. In the unit circle, the formula becomes $a = \theta(1)$ or $a = \theta$. This means that a central angle and its subtended arc on the unit circle have the same numerical value.

You can use the function $P(\theta) = (x, y)$ to link the arc length, θ, of a central angle in the unit circle to the coordinates, (x, y), of the point of intersection of the terminal arm and the unit circle.

If you join $P(\theta)$ to the origin, you create an angle θ in standard position. Now, θ radians is the central angle and the arc length is θ units.

Function P takes real-number values for the central angle or the arc length on the unit circle and matches them with specific points. For example, if $\theta = \pi$, the point is $(-1, 0)$. Thus, you can write $P(\pi) = (-1, 0)$.



Example 3

Multiples of $\frac{\pi}{3}$ on the Unit Circle

a) On a diagram of the unit circle, show the integral multiples of $\frac{\pi}{3}$ in the interval $0 \le \theta \le 2\pi$.

b) What are the coordinates for each point $P(\theta)$ in part a)?

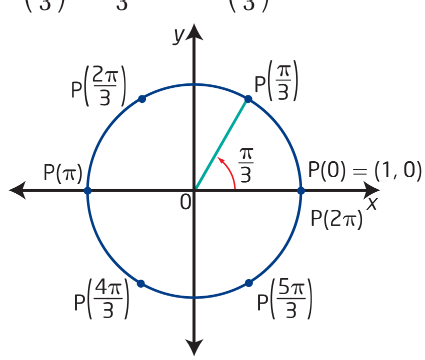
c) Identify any patterns you see in the coordinates of the points.

Solution

a) This is essentially a counting problem using $\frac{\pi}{3}$.

Multiples of $\frac{\pi}{3}$ in the interval $0 \le \theta \le 2\pi$ are

$0\left(\frac{\pi}{3}\right) = 0$, $1\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$, $2\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}$, $3\left(\frac{\pi}{3}\right) = \pi$, $4\left(\frac{\pi}{3}\right) = \frac{4\pi}{3}$, $5\left(\frac{\pi}{3}\right) = \frac{5\pi}{3}$, and $6\left(\frac{\pi}{3}\right) = 2\pi$.

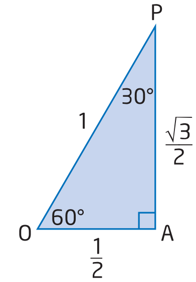


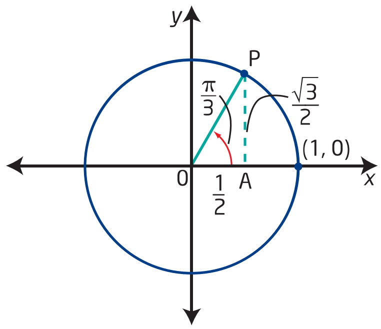
Why must you show only the multiples in one positive rotation in the unit circle?

b) Recall that a 30°-60°-90° triangle has sides in the ratio $1:\sqrt{3}:2$ or $\frac{1}{2}:\frac{\sqrt{3}}{2}:1$.

Place $\triangle$POA in the unit circle as shown.

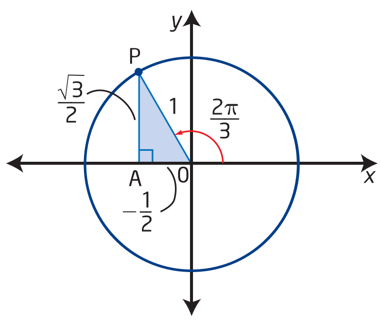
Why is the 30°-60°-90° triangle used?





Why are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ the coordinates of $P\left(\frac{\pi}{3}\right)$?

$\triangle$POA could be placed in the second quadrant with O at the origin and OA along the *x*-axis as shown. This gives $P\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



Why is the *x*-coordinate negative?

What transformation could be used to move $\triangle$POA from quadrant I to quadrant II?

Continue, placing $\triangle$POA in quadrants III and IV to find the coordinates of $P\left(\frac{4\pi}{3}\right)$ and $P\left(\frac{5\pi}{3}\right)$. Then, the coordinates of point P corresponding to angles that are multiples of $\frac{\pi}{3}$ are

$P(0) = P(2\pi) = (1, 0)$	$P(\pi) = (-1, 0)$	$P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$P\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$P\left(\frac{4\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

c) Some patterns are:

- The points corresponding to angles that are multiples of $\frac{\pi}{3}$ that cannot be simplified, for example, $P\left(\frac{\pi}{3}\right)$, $P\left(\frac{2\pi}{3}\right)$, $P\left(\frac{4\pi}{3}\right)$, and $P\left(\frac{5\pi}{3}\right)$, have the same coordinates except for their signs.
- Any points where θ reduces to a multiple of π, for example, P(0), $P\left(\frac{3\pi}{3}\right) = P(\pi)$, and $P\left(\frac{6\pi}{3}\right) = P(2\pi)$, fall on an axis.

Your Turn

a) On a diagram of the unit circle, show all the integral multiples of $\frac{\pi}{6}$ in the interval $0 \le \theta < 2\pi$.

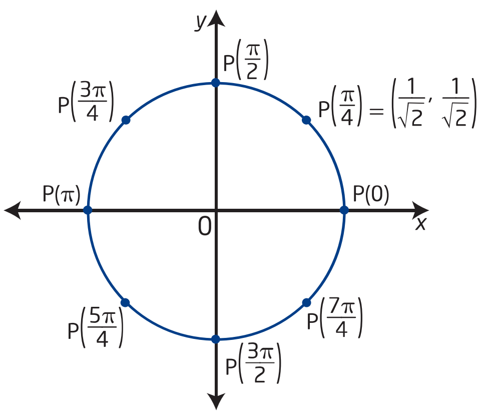
b) Label the coordinates for each point P(θ) on your diagram.

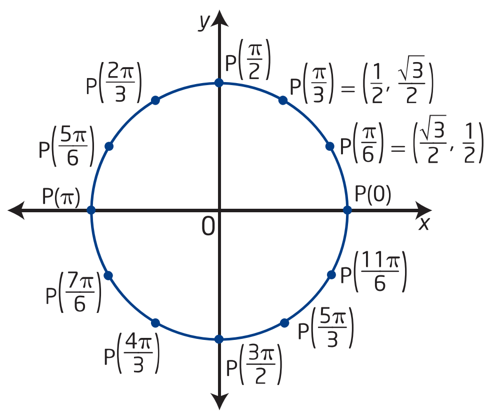
c) Describe any patterns you see in the coordinates of the points.

Key Ideas

- The equation for the unit circle is $x^2 + y^2 = 1$. It can be used to determine whether a point is on the unit circle or to determine the value of one coordinate given the other. The equation for a circle with centre at (0, 0) and radius r is $x^2 + y^2 = r^2$.
- On the unit circle, the measure in radians of the central angle and the arc subtended by that central angle are numerically equivalent.
- Some of the points on the unit circle correspond to exact values of the special angles learned previously.
- You can use patterns to determine coordinates of points. For example, the numerical value of the coordinates of points on the unit circle change to their opposite sign every $\frac{1}{2}$ rotation.

  If $P(\theta) = (a, b)$ is in quadrant I, then both a and b are positive. $P(\theta + \pi)$ is in quadrant III. Its coordinates are $(-a, -b)$, where $a > 0$ and $b > 0$.





Check Your Understanding

Practise

1. Determine the equation of a circle with centre at the origin and radius
   a) 4 units
   b) 3 units
   c) 12 units
   d) 2.6 units

2. Is each point on the unit circle? How do you know?
   a) $\left(-\frac{3}{4}, \frac{1}{4}\right)$
   b) $\left(\frac{\sqrt{5}}{8}, \frac{7}{8}\right)$
   c) $\left(-\frac{5}{13}, \frac{12}{13}\right)$
   d) $\left(\frac{4}{5}, -\frac{3}{5}\right)$
   e) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
   f) $\left(\frac{\sqrt{7}}{4}, \frac{3}{4}\right)$

3. Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions. Draw a diagram to support your answer.

a) $\left(\frac{1}{4}, y\right)$ in quadrant I

b) $\left(x, \frac{2}{3}\right)$ in quadrant II

c) $\left(-\frac{7}{8}, y\right)$ in quadrant III

d) $\left(x, -\frac{5}{7}\right)$ in quadrant IV

e) $\left(x, \frac{1}{3}\right)$, where $x < 0$

f) $\left(\frac{12}{13}, y\right)$, not in quadrant I

4. If $P(\theta)$ is the point at the intersection of the terminal arm of angle θ and the unit circle, determine the exact coordinates of each of the following.

a) $P(\pi)$

b) $P\left(-\frac{\pi}{2}\right)$

c) $P\left(\frac{\pi}{3}\right)$

d) $P\left(-\frac{\pi}{6}\right)$

e) $P\left(\frac{3\pi}{4}\right)$

f) $P\left(-\frac{7\pi}{4}\right)$

g) $P(4\pi)$

h) $P\left(\frac{5\pi}{2}\right)$

i) $P\left(\frac{5\pi}{6}\right)$

j) $P\left(-\frac{4\pi}{3}\right)$

5. Identify a measure for the central angle θ in the interval $0 \leq \theta < 2\pi$ such that $P(\theta)$ is the given point.

a) $(0, -1)$

b) $(1, 0)$

c) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

d) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

e) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

f) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

g) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

h) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

i) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

j) $(-1, 0)$

6. Determine one positive and one negative measure for θ if $P(\theta) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Apply

7. Draw a diagram of the unit circle.

a) Mark two points, $P(\theta)$ and $P(\theta + \pi)$, on your diagram. Use measurements to show that these points have the same coordinates except for their signs.

b) Choose a different quadrant for the original point, $P(\theta)$. Mark it and $P(\theta + \pi)$ on your diagram. Is the result from part a) still true?

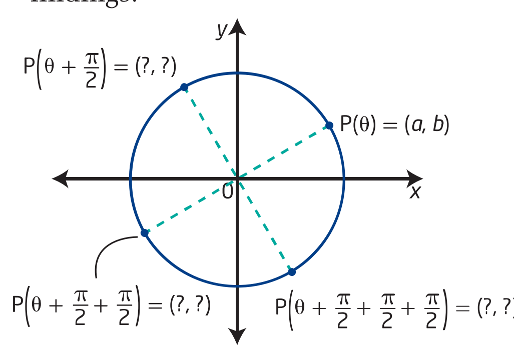
8. **MINI LAB** Determine the pattern in the coordinates of points that are $\frac{1}{4}$ rotation apart on the unit circle.

Step 1 Start with the points $P(0) = (1, 0)$, $P\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Show these points on a diagram.

Step 2 Move $+\frac{1}{4}$ rotation from each point. Determine each new point and its coordinates. Show these points on your diagram from step 1.

Step 3 Move $-\frac{1}{4}$ rotation from each original point. Determine each new point and its coordinates. Mark these points on your diagram.

Step 4 How do the values of the x-coordinates and y-coordinates of points change with each quarter-rotation? Make a copy of the diagram and complete the coordinates to summarize your findings.



9. Use the diagram below to help answer these questions.

a) What is the equation of this circle?

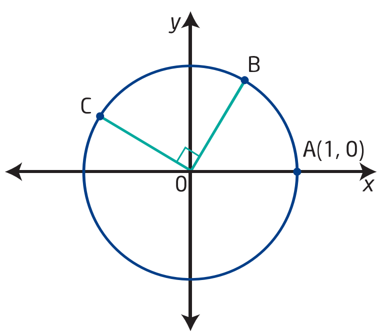
b) If the coordinates of C are $\left(-\frac{2}{3}, \frac{\sqrt{5}}{3}\right)$, what are the coordinates of B?

c) If the measure of $\overset{\frown}{AB}$ is θ, what is an expression for the measure of $\overset{\frown}{AC}$?

Note: $\overset{\frown}{AB}$ means the arc length from A to B.

d) Let $P(\theta) = B$. In which quadrant is $P\left(\theta - \frac{\pi}{2}\right)$?

e) What are the maximum and minimum values for either the *x*-coordinates or *y*-coordinates of points on the unit circle?



10. Mya claims that every value of *x* between 0 and 1 can be used to find the coordinates of a point on the unit circle in quadrant I.

a) Do you agree with Mya? Explain.

b) Mya showed the following work to find the *y*-coordinate when $x = 0.807$.

$y = 1 - (0.807)^2$
$= 0.348\ 751$

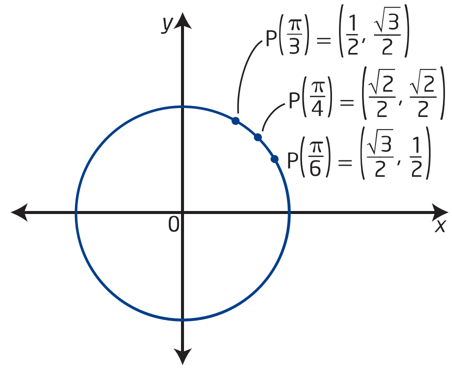
The point on the unit circle is (0.807, 0.348 751).

How can you check Mya's answer? Is she correct? If not, what is the correct answer?

c) If $y = 0.2571$, determine *x* so the point is on the unit circle and in the first quadrant.

11. Wesley enjoys tricks and puzzles. One of his favourite tricks involves remembering the coordinates for $P\left(\frac{\pi}{3}\right)$, $P\left(\frac{\pi}{4}\right)$, and $P\left(\frac{\pi}{6}\right)$. He will not tell you his trick. However, you can discover it for yourself.

a) Examine the coordinates shown on the diagram.



b) What do you notice about the denominators?

c) What do you notice about the numerators of the *x*-coordinates? Compare them with the numerators of the *y*-coordinates. Why do these patterns make sense?

d) Why are square roots involved?

e) Explain this memory trick to a partner.

12. a) Explain, with reference to the unit circle, what the interval $-2\pi \le \theta < 4\pi$ represents.

b) Use your explanation to determine all values for θ in the interval $-2\pi \le \theta < 4\pi$ such that $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

c) How do your answers relate to the word "coterminal"?

13. If $P(\theta) = \left(-\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$, determine the following.

a) What does $P(\theta)$ represent? Explain using a diagram.

b) In which quadrant does θ terminate?

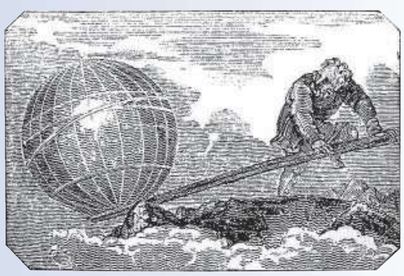
c) Determine the coordinates of $P\left(\theta + \frac{\pi}{2}\right)$.

d) Determine the coordinates of $P\left(\theta - \frac{\pi}{2}\right)$.

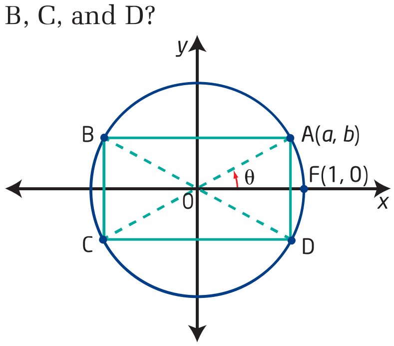
14. In ancient times, determining the perimeter and area of a circle were considered major mathematical challenges. One of Archimedes' greatest contributions to mathematics was his method for approximating π. Now, it is your turn to be a mathematician. Using a unit circle diagram, show the difference between π units and π square units.

Did You Know?

Archimedes was a Greek mathematician, physicist, inventor, and astronomer who lived from 287 BCE–212 BCE. He died in the Roman siege of Syracuse. He is considered one of the greatest mathematicians of all time. He correctly determined the value of π as being between $\frac{22}{7}$ and $\frac{223}{71}$ and proved the area of a circle to be πr^2, where r is the radius.



15. a) In the diagram, A has coordinates (a, b). ABCD is a rectangle with sides parallel to the axes. What are the coordinates of B, C, and D?



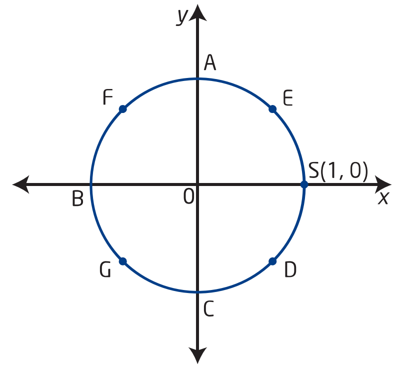
b) ∠FOA = θ, and A, B, C, and D lie on the unit circle. Through which point will the terminal arm pass for each angle? Assume all angles are in standard position.

i) $\theta + \pi$ **ii)** $\theta - \pi$

iii) $-\theta + \pi$ **iv)** $-\theta - \pi$

c) How are the answers in part b) different if θ is given as the measure of arc FA?

16. Use the unit circle diagram to answer the following questions. Points E, F, G, and D are midway between the axes.



a) What angle of rotation creates arc SG? What is the arc length of SG?

b) Which letter on the diagram corresponds to $P\left(\frac{13\pi}{2}\right)$? Explain your answer fully so someone not taking this course would understand. Use a diagram and a written explanation.

c) Between which two points would you find P(5)? Explain.

Extend

17. a) Determine the coordinates of all points where the line represented by $y = -3x$ intersects the unit circle. Give your answers as exact values in simplest form.

b) If one of the points is labelled $P(\theta + \pi)$, draw a diagram and show at least two values for θ. Explain what θ represents.

18. a) P(θ) lies at the intersection of the unit circle and the line joining A(5, 2) to the origin. Use your knowledge of similar triangles and the unit circle to determine the exact coordinates of P(θ).

b) Determine the radius of a larger circle with centre at the origin and passing through point A.

c) Write the equation for this larger circle.