Example 1

Convert Between Degree and Radian Measure

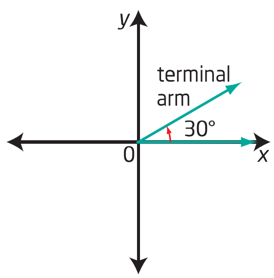
Draw each angle in standard position. Change each degree measure to radian measure and each radian measure to degree measure. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

a) $30°$ **b)** $-120°$

c) $\frac{5\pi}{4}$ **d)** 2.57

Solution

a)



An angle in standard position has its centre at the origin and its initial arm along the positive *x*-axis.

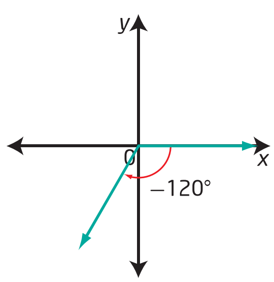
In which direction are positive angles measured?

Unitary Method

$$360° = 2\pi$$
$$1° = \frac{2\pi}{360}$$
$$= \frac{\pi}{180}$$
$$30° = 30\left(\frac{\pi}{180}\right)$$
$$= \frac{\pi}{6}$$
$$\approx 0.52$$

$\frac{\pi}{6}$ is an exact value.

b)



Why is the angle drawn using a clockwise rotation?

Proportion Method

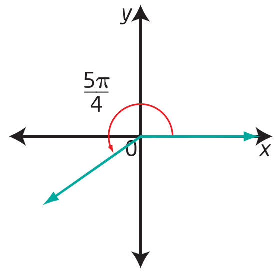
$$180° = \pi$$
$$\frac{-120°}{180°} = \frac{x}{\pi}$$
$$x = \frac{-120\pi}{180}$$
$$= -\frac{2\pi}{3}$$
$$\approx -2.09$$

So, $-120°$ is equivalent to $-\frac{2\pi}{3}$ or approximately -2.09.

c) π is $\frac{1}{2}$ rotation.

$\frac{\pi}{4}$ is $\frac{1}{8}$ rotation.

So $\frac{5\pi}{4}$ terminates in the third quadrant.



Unit Analysis

$$\frac{5\pi}{4} = \left(\frac{5\pi}{4}\right)\left(\frac{180°}{\pi}\right)$$
$$= \frac{5(180°)}{4}$$
$$= 225°$$

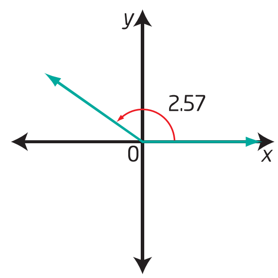
$\frac{5\pi}{4}$ is equivalent to 225°.

Why does $\left(\frac{180°}{\pi}\right)$ have value 1?

d) π (approximately 3.14) is $\frac{1}{2}$ rotation.

$\frac{\pi}{2}$ (approximately 1.57) is $\frac{1}{4}$ rotation.

2.57 is between 1.57 and 3.14, so it terminates in the second quadrant.



Unitary Method

$$\pi = 180°$$
$$1 = \frac{180°}{\pi}$$
$$2.57$$
$$= 2.57\left(\frac{180°}{\pi}\right)$$
$$= \frac{462.6°}{\pi}$$
$$\approx 147.25°$$

Proportion Method

$$\frac{x}{2.57} = \frac{180°}{\pi}$$
$$x = 2.57\left(\frac{180°}{\pi}\right)$$
$$x = \frac{462.6°}{\pi}$$
$$x \approx 147.25°$$

Unit Analysis

$$2.57$$
$$= 2.57\left(\frac{180°}{\pi}\right)$$
$$= \frac{462.6°}{\pi}$$
$$\approx 147.25°$$

2.57 is equivalent to $\frac{462.6°}{\pi}$ or approximately 147.25°.

Your Turn

Draw each angle in standard position. Change each degree measure to radians and each radian measure to degrees. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

a) $-270°$

b) 150°

c) $\frac{7\pi}{6}$

d) -1.2

Did You Know?

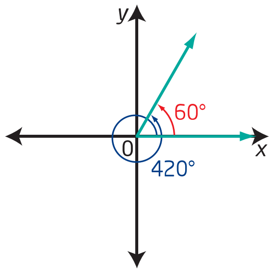
Most scientific and graphing calculators can calculate using angle measures in both degrees and radians. Find out how to change the mode on your calculator.

Coterminal Angles

coterminal angles

- angles in standard position with the same terminal arms
- may be measured in degrees or radians
- $\frac{\pi}{4}$ and $\frac{9\pi}{4}$ are coterminal angles, as are 40° and −320°

When you sketch an angle of 60° and an angle of 420° in standard position, the terminal arms coincide. These are **coterminal angles**.



Example 2

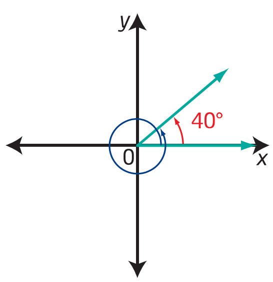
Identify Coterminal Angles

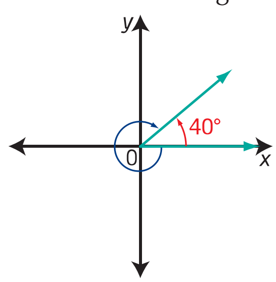
Determine one positive and one negative angle measure that is coterminal with each angle. In which quadrant does the terminal arm lie?

a) 40° **b)** −430° **c)** $\frac{8\pi}{3}$

Solution

a) The terminal arm is in quadrant I.
To locate coterminal angles, begin on the terminal arm of the given angle and rotate in a positive or negative direction until the new terminal arm coincides with that of the original angle.



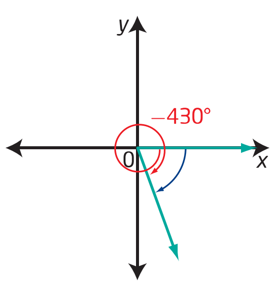


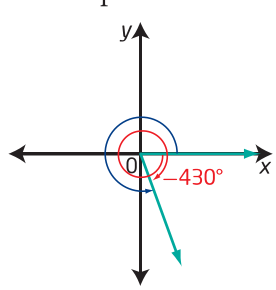
$40° + 360° = 400°$ $40° + (-360°) = -320°$

Two angles coterminal with 40° are 400° and −320°.

What other answers are possible?

b) The terminal arm of −430° is in quadrant IV.



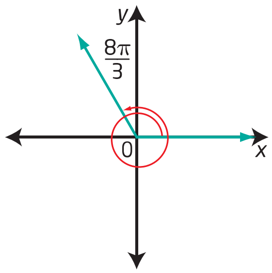


$-430° + 360° = -70°$ $-430° + 720° = 290°$

Two angles coterminal with −430° are 290° and −70°.

The reference angle is 70°.

c)



$\frac{8\pi}{3} = \frac{6\pi}{3} + \frac{2\pi}{3}$

So, the angle is one full rotation (2π) plus $\frac{2\pi}{3}$.

The terminal arm is in quadrant II.

There are 2π or $\frac{6\pi}{3}$ in one full rotation.

Counterclockwise one full rotation: $\frac{8\pi}{3} + \frac{6\pi}{3} = \frac{14\pi}{3}$

Clockwise one full rotation: $\frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$

Clockwise two full rotations: $\frac{8\pi}{3} - \frac{12\pi}{3} = -\frac{4\pi}{3}$

Two angles coterminal with $\frac{8\pi}{3}$ are $\frac{2\pi}{3}$ and $-\frac{4\pi}{3}$.

Your Turn

For each angle in standard position, determine one positive and one negative angle measure that is coterminal with it.

a) $270°$ **b)** $-\frac{5\pi}{4}$ **c)** $740°$

Coterminal Angles in General Form

By adding or subtracting multiples of one full rotation, you can write an infinite number of angles that are coterminal with any given angle.

For example, some angles that are coterminal with 40° are

$40° + (360°)(1) = 400°$ $40° - (360°)(1) = -320°$

$40° + (360°)(2) = 760°$ $40° - (360°)(2) = -680°$

In general, the angles coterminal with 40° are $40° \pm (360°)n$, where n is any natural number.

Some angles coterminal with $\frac{2\pi}{3}$ are

$$\frac{2\pi}{3} + 2\pi(1) = \frac{2\pi}{3} + \frac{6\pi}{3} = \frac{8\pi}{3} \qquad \frac{2\pi}{3} - 2\pi(1) = \frac{2\pi}{3} - \frac{6\pi}{3} = -\frac{4\pi}{3}$$

$$\frac{2\pi}{3} + 2\pi(2) = \frac{2\pi}{3} + \frac{12\pi}{3} = \frac{14\pi}{3} \qquad \frac{2\pi}{3} - 2\pi(2) = \frac{2\pi}{3} - \frac{12\pi}{3} = -\frac{10\pi}{3}$$

In general, the angles coterminal with $\frac{2\pi}{3}$ are $\frac{2\pi}{3} \pm 2\pi n$, where n is any natural number.

Any given angle has an infinite number of angles coterminal with it, since each time you make one full rotation from the terminal arm, you arrive back at the same terminal arm. Angles coterminal with any angle θ can be described using the expression

$\theta \pm (360°)n$ or $\theta \pm 2\pi n$,

where n is a natural number. This way of expressing an answer is called the **general form**.

general form

- an expression containing parameters that can be given specific values to generate any answer that satisfies the given information or situation
- represents all possible cases

Example 3

Express Coterminal Angles in General Form

a) Express the angles coterminal with 110° in general form. Identify the angles coterminal with 110° that satisfy the domain $-720° \le \theta < 720°$.

b) Express the angles coterminal with $\frac{8\pi}{3}$ in general form. Identify the angles coterminal with $\frac{8\pi}{3}$ in the domain $-4\pi \le \theta < 4\pi$.

Solution

a) Angles coterminal with 110° occur at 110° $\pm (360°)n$, $n \in \mathrm{N}$.

Substitute values for n to determine these angles.

n	1	2	3
$110° - (360°)n$	$-250°$	$-610°$	$-970°$
$110° + (360°)n$	470°	830°	1190°

From the table, the values that satisfy the domain $-720° \le \theta < 720°$ are $-610°$, $-250°$, and 470°. These angles are coterminal.

b) $\frac{8\pi}{3} \pm 2\pi n$, $n \in \mathrm{N}$, represents all angles coterminal with $\frac{8\pi}{3}$.
Substitute values for n to determine these angles.

n	1	2	3	4
$\frac{8\pi}{3} - 2\pi n$	$\frac{2\pi}{3}$	$-\frac{4\pi}{3}$	$-\frac{10\pi}{3}$	$-\frac{16\pi}{3}$
$\frac{8\pi}{3} + 2\pi n$	$\frac{14\pi}{3}$	$\frac{20\pi}{3}$	$\frac{26\pi}{3}$	$\frac{32\pi}{3}$

The angles in the domain $-4\pi \le \theta < 4\pi$ that are coterminal are $-\frac{10\pi}{3}$, $-\frac{4\pi}{3}$, and $\frac{2\pi}{3}$.

Why is $-\frac{16\pi}{3}$ not an acceptable answer?

Your Turn

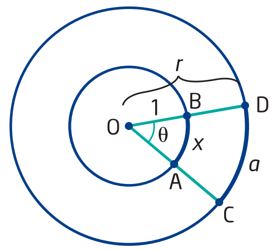
Write an expression for all possible angles coterminal with each given angle. Identify the angles that are coterminal that satisfy $-360° \le \theta < 360°$ or $-2\pi \le \theta < 2\pi$.

a) $-500°$ **b)** 650° **c)** $\frac{9\pi}{4}$

Arc Length of a Circle

All arcs that subtend a right angle $\left(\frac{\pi}{2}\right)$ have the same central angle, but they have different arc lengths depending on the radius of the circle. The arc length is proportional to the radius. This is true for any central angle and related arc length.

Consider two concentric circles with centre O. The radius of the smaller circle is 1, and the radius of the larger circle is r. A central angle of θ radians is subtended by arc AB on the smaller circle and arc CD on the larger one. You can write the following proportion, where x represents the arc length of the smaller circle and a is the arc length of the larger circle.



$$\frac{a}{x} = \frac{r}{1}$$
$$a = xr \qquad ①$$

Consider the circle with radius 1 and the sector with central angle θ. The ratio of the arc length to the circumference is equal to the ratio of the central angle to one full rotation.

$$\frac{x}{2\pi r} = \frac{\theta}{2\pi}$$

Why is $r = 1$?

$$x = \left(\frac{\theta}{2\pi}\right)2\pi(1)$$
$$x = \theta$$

Substitute $x = \theta$ in ①.

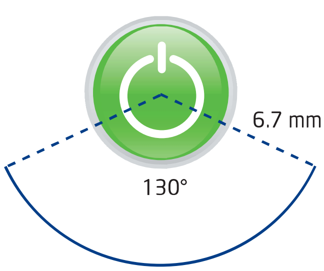
$$a = \theta r$$

This formula, $a = \theta r$, works for any circle, provided that θ is measured in radians and both a and r are measured in the same units.

Example 4

Determine Arc Length in a Circle

Rosemarie is taking a course in industrial engineering. For an assignment, she is designing the interface of a DVD player. In her plan, she includes a decorative arc below the on/off button. The arc has central angle 130° in a circle with radius 6.7 mm. Determine the length of the arc, to the nearest tenth of a millimetre.



Solution

Method 1: Convert to Radians and Use the Formula $a = \theta r$

Convert the measure of the central angle to radians before using the formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the length of the radius.

$$180° = \pi$$
$$1° = \frac{\pi}{180}$$
$$130° = 130\left(\frac{\pi}{180}\right)$$
$$= \frac{13\pi}{18}$$

$$a = \theta r$$
$$= \left(\frac{13\pi}{18}\right)(6.7)$$
$$= \frac{87.1\pi}{18}$$
$$= 15.201\ldots$$

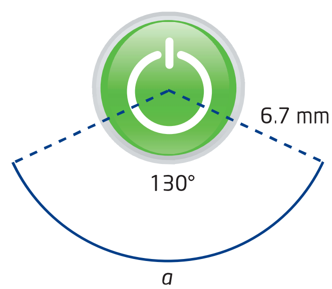
Why is it important to use exact values throughout the calculation and only convert to decimal fractions at the end?

The arc length is 15.2 mm, to the nearest tenth of a millimetre.

Method 2: Use a Proportion

Let a represent the arc length.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{central angle}}{\text{full rotation}}$$
$$\frac{a}{2\pi(6.7)} = \frac{130°}{360°}$$
$$a = \frac{2\pi(6.7)130°}{360°}$$
$$= 15.201\ldots$$



The arc length is 15.2 mm, to the nearest tenth of a millimetre.

Your Turn

If a represents the length of an arc of a circle with radius r, subtended by a central angle of θ, determine the missing quantity. Give your answers to the nearest tenth of a unit.

a) $r = 8.7$ cm, $\theta = 75°$, $a = ■$ cm

b) $r = ■$ mm, $\theta = 1.8$, $a = 4.7$ mm

c) $r = 5$ m, $a = 13$ m, $\theta = ■$

Key Ideas

- Angles can be measured using different units, including degrees and radians.
- An angle measured in one unit can be converted to the other unit using the relationships 1 full rotation $= 360° = 2\pi$.
- An angle in standard position has its vertex at the origin and its initial arm along the positive x-axis.
- Angles that are coterminal have the same initial arm and the same terminal arm.
- An angle θ has an infinite number of angles that are coterminal expressed by $\theta \pm (360°)n$, $n \in N$, in degrees, or $\theta \pm 2\pi n$, $n \in N$, in radians.
- The formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the length of the radius, can be used to determine any of the variables given the other two, as long as a and r are in the same units.

Check Your Understanding

Practise

1. For each angle, indicate whether the direction of rotation is clockwise or counterclockwise.

a) -4π **b)** $750°$
c) $-38.7°$ **d)** 1

2. Convert each degree measure to radians. Write your answers as exact values. Sketch the angle and label it in degrees and in radians.

a) $30°$ **b)** $45°$
c) $-330°$ **d)** $520°$
e) $90°$ **f)** $21°$

3. Convert each degree measure to radians. Express your answers as exact values and as approximate measures, to the nearest hundredth of a radian.

a) $60°$ **b)** $150°$
c) $-270°$ **d)** $72°$
e) $-14.8°$ **f)** $540°$

4. Convert each radian measure to degrees. Express your answers as exact values and as approximate measures, to the nearest tenth of a degree, if necessary.

a) $\frac{\pi}{6}$ **b)** $\frac{2\pi}{3}$
c) $-\frac{3\pi}{8}$ **d)** $-\frac{5\pi}{2}$
e) 1 **f)** 2.75

5. Convert each radian measure to degrees. Express your answers as exact values and as approximate measures, to the nearest thousandth.

a) $\frac{2\pi}{7}$ **b)** $\frac{7\pi}{13}$
c) $\frac{2}{3}$ **d)** 3.66
e) -6.14 **f)** -20

6. Sketch each angle in standard position. In which quadrant does each angle terminate?

a) 1 **b)** $-225°$
c) $\frac{17\pi}{6}$ **d)** $650°$
e) $-\frac{2\pi}{3}$ **f)** $-42°$

7. Determine one positive and one negative angle coterminal with each angle.

a) $72°$
b) $\frac{3\pi}{4}$
c) $-120°$
d) $\frac{11\pi}{2}$
e) $-205°$
f) 7.8

8. Determine whether the angles in each pair are coterminal. For one pair of angles, explain how you know.

a) $\frac{5\pi}{6}, \frac{17\pi}{6}$
b) $\frac{5\pi}{2}, -\frac{9\pi}{2}$
c) $410°, -410°$
d) $227°, -493°$

9. Write an expression for all of the angles coterminal with each angle. Indicate what your variable represents.

a) $135°$
b) $-\frac{\pi}{2}$
c) $-200°$
d) 10

10. Draw and label an angle in standard position with negative measure. Then, determine an angle with positive measure that is coterminal with your original angle. Show how to use a general expression for coterminal angles to find the second angle.

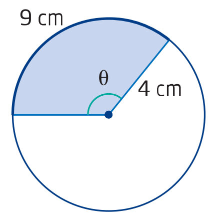
11. For each angle, determine all angles that are coterminal in the given domain.

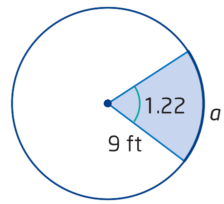
a) $65°, 0° \leq \theta < 720°$
b) $-40°, -180° \leq \theta < 360°$
c) $-40°, -720° \leq \theta < 720°$
d) $\frac{3\pi}{4}, -2\pi \leq \theta < 2\pi$
e) $-\frac{11\pi}{6}, -4\pi \leq \theta < 4\pi$
f) $\frac{7\pi}{3}, -2\pi \leq \theta < 4\pi$
g) $2.4, -2\pi \leq \theta < 2\pi$
h) $-7.2, -4\pi \leq \theta < 2\pi$

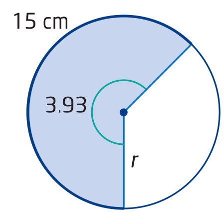
12. Determine the arc length subtended by each central angle. Give answers to the nearest hundredth of a unit.

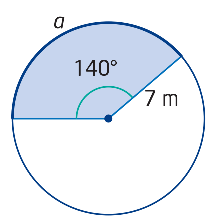
a) radius 9.5 cm, central angle 1.4
b) radius 1.37 m, central angle 3.5
c) radius 7 cm, central angle 130°
d) radius 6.25 in., central angle 282°

13. Use the information in each diagram to determine the value of the variable. Give your answers to the nearest hundredth of a unit.

a) 

b) 

c) 

d) 

Apply

14. A rotating water sprinkler makes one revolution every 15 s. The water reaches a distance of 5 m from the sprinkler.

a) What is the arc length of the sector watered when the sprinkler rotates through $\frac{5\pi}{3}$? Give your answer as both an exact value and an approximate measure, to the nearest hundredth.
b) Show how you could find the area of the sector watered in part a).
c) What angle does the sprinkler rotate through in 2 min? Express your answer in radians and degrees.

15. Angular velocity describes the rate of change in a central angle over time. For example, the change could be expressed in revolutions per minute (rpm), radians per second, degrees per hour, and so on. All that is required is an angle measurement expressed over a unit of time.

a) Earth makes one revolution every 24 h. Express the angular velocity of Earth in three other ways.

b) An electric motor rotates at 1000 rpm. What is this angular velocity expressed in radians per second?

c) A bicycle wheel completes 10 revolutions every 4 s. Express this angular velocity in degrees per minute.

16. Skytrek Adventure Park in Revelstoke, British Columbia, has a sky swing. Can you imagine a 170-ft flight that takes riders through a scary pendulum swing? At one point you are soaring less than 10 ft from the ground at speeds exceeding 60 mph.

a) The length of the cable is 72 ft and you travel on an arc of length 170 ft on one particular swing. What is the measure of the central angle? Give your answer in radians, to the nearest hundredth.

b) What is the measure of the central angle from part a), to the nearest tenth of a degree?

17. Copy and complete the table by converting each angle measure to its equivalent in the other systems. Round your answers to the nearest tenth where necessary.

	Revolutions	Degrees	Radians
a)	1 rev		
b)		270°	
c)			$\frac{5\pi}{6}$
d)			−1.7
e)		−40°	
f)	0.7 rev		
g)	−3.25 rev		
h)		460°	
i)			$-\frac{3\pi}{8}$

18. Joran and Jasmine are discussing expressions for the general form of coterminal angles of 78°. Joran claims the answer must be expressed as $78° + (360°)n$, $n \in I$. Jasmine indicates that although Joran's expression is correct, another answer is possible. She prefers $78° \pm k(360°)$, $k \in N$, where N represents positive integers. Who is correct? Why?

19. The gradian (grad) is another unit of angle measure. It is defined as $\frac{1}{400}$ of a revolution, so one full rotation contains 400 grads.

a) Determine the number of gradians in 50°.

b) Describe a process for converting from degree measure to gradians and vice versa.

c) Identify a possible reason that the gradian was created.

Did You Know?

Gradians originated in France in the 1800s. They are still used in some engineering work.